Quasi-Normal Scale Elimination (QNSE) theory of anisotropic turbulence and waves in flows with stable stratification

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Introduction

• The aim is to develop a theory that systematically includes anisotropic turbulence and internal waves.

• **The difficulty:** turbulence is a nonlinear, multi-scale, stochastic phenomenon. Analytical theories exist for simplest flows that are locally isotropic. **Geophysical flows are anisotropic with waves.**

• Reynolds averaging does not differentiate between scales and does not discern contributions from different processes. Reynolds stress closures employ the concept of “invariant modeling” and are not flexible enough.

• **Spectral approach is more suitable**
The Quasi-Normal Scale Elimination (QNSE) theory of turbulence with stratification

We consider fully 3D turbulent flow with imposed vertical temperature gradient \( \frac{d\Theta}{dz} \).

Governing equations in Boussinesq approximation:

**momentum** \[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla)\mathbf{u} - \alpha g T \hat{\mathbf{e}}_3 = \nu_0 \nabla^2 \mathbf{u} - \frac{\nabla P}{\rho} + \mathbf{f}_0
\]

**temperature** \[
\frac{\partial T}{\partial t} + (\mathbf{u} \nabla)T + \frac{d\Theta}{dz} u_3 = \kappa_0 \nabla^2 T
\]

**continuity** \[
\nabla \mathbf{u} = 0
\]

In linear approximation this system supports gravity waves with Brunt-Vaisala frequency

\[
N \equiv \left( \alpha g \frac{d\Theta}{dz} \right)^{1/2}
\]
Fourier-transformed velocity and temperature equations

Eliminate pressure using continuity equation; obtain momentum equation in a self-contained form using formal solution to the temperature equation:

\[
\begin{align*}
    u_\alpha(\hat{k}) &= G_{\alpha\beta}(\hat{k}) \left[ \phi_\beta^T(\hat{k}) - \frac{i}{2} P_{\beta\mu\nu}(\hat{k}) \int u_\mu(\hat{q}) u_\nu(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^4} \right]
    \quad \text{where } f_\beta(\hat{k})
\end{align*}
\]

\[
\begin{align*}
    T(\hat{k}) &= G_T(\hat{k}) \left\{ -\frac{d\Theta}{dz} u_3(\hat{k}) - ik_\alpha \int u_\alpha(\hat{q}) T(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^4} \right\}
    \quad \text{where } f_T(\hat{k})
\end{align*}
\]

Velocity Green function becomes tensorial:

\[
    G_{\alpha\beta}(\mathbf{k}, \omega) = G(\mathbf{k}, \omega) \left[ \delta_{\alpha\beta} + A(\mathbf{k}, \omega) P_{\alpha\beta}(\mathbf{k}) \delta_{\beta3} \right]
\]

where \( G(\mathbf{k}, \omega) = \left[ -i \omega + \nu_h(k) k_h^2 + \nu_z(k) k_3^2 \right]^{-1} \)

\( \nu_h, \nu_z \) - horizontal and vertical eddy viscosities, \( P_{\alpha\beta} \) - projection operator

\[
    A(\mathbf{k}, \omega) = -\frac{N^2}{\left( -i \omega + \nu k_h^2 + \nu_z k_3^2 \right) \left( -i \omega + \kappa k_h^2 + \kappa_z k_3^2 \right) + N^2 \sin^2 \phi}
\]

\( \phi \) is the angle between \( \mathbf{k} \) and the vertical,

\[
    G_T(\mathbf{k}, \omega) = \left[ -i \omega + \kappa_h(k) k_h^2 + \kappa_z(k) k_3^2 \right]^{-1}
\]

is temperature Green function, \( \kappa_h \) and \( \kappa_z \) are horizontal and vertical eddy diffusivities
Quasi-Normal Scale Elimination Model (QNSE)

As in other quasi-normal models, we seek the solution in the form

\[ u_\alpha(k, \omega) = G_{\alpha\beta}(k, \omega) f_\beta(k, \omega) \]  
\[ T(k, \omega) = G_T(k, \omega) \left( f_T(k, \omega) - \frac{d}{dz} u_3(k, \omega) \right) \]

In other words: mapping onto quasi-Gaussian fields governed by the Langevin equations

\[ f_\alpha(k, \omega) \] is a stochastic force representing **stirring of a given velocity mode by all other modes**; postulated as quasi-Gaussian, solenoidal, homogeneous in space and time:

\[ \langle f_\alpha(\omega, k) f_\beta(\omega', k') \rangle \propto e^{-3} P_{\alpha\beta}(k) \delta(\omega + \omega') \delta(k + k') \]

**Effective viscosities and diffusivities** in Green functions describe **damping of a mode by nonlinear interactions with all other modes**.

Due to anisotropy, viscosities and diffusivities are different in the vertical and horizontal directions.

Our goal is to calculate the mapping parameters, \( v_h, v_z, \kappa_h, \kappa_z \).
Quasi-Normal Scale Elimination (QNSE) method

Central problem is treatment of nonlinearity. Perturbative solution based on expansion parameter Re?

It is strongly divergent!

General idea: Re is small for smallest scales of motion =>

• Derive perturbative solution for these small scales
• Using this solution and assumption of Quasi-Gaussianity perform averaging over infinitesimal band of small scales. Compute corrections to “effective” or “eddy” viscosity and heat diffusivity. Viscosity increases; Re for the next band remains small
• Repeat the above procedure for the next band of smallest scales.

Partial scale elimination yields a subgrid-scale model for LES; complete scale elimination yields eddy viscosities and eddy diffusivities for RANS (Reynolds-average Navier-Stokes) models.
Corrections to viscosities and diffusivities

Diagrammatic technique

\[ u_\alpha(\hat{k}) = \]
\[ G_{\alpha\beta}(\hat{k}) = \]
\[ f_{\beta}^{0}(\hat{k}) = \times \]

Momentum equation:
Subdivision into “slow” (large) and “fast” (small) modes

Additional iteration (non-conventional, slow modes are iterated)

Replace the fast velocity modes by their Langevin equations and ensemble-average over the fast modes

After the averaging only diagrams 5 and 9 remain.
Assumption of quasi-Gaussianity means that triple correlators (6), (10) vanish
**Corrections to viscosities and diffusivities**

**Correction to viscose term:**

\[ 4 \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \]

**Correction to diffusive term**

\[ \left( \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) \]

Analytical expressions for the corrections are given by

\[ \Delta G_{\alpha\beta}^{-1}(\omega,k,k_3) = P_{\alpha\beta\varphi}(k) \int P_{\nu\sigma\beta}(k-q) U_{\mu\sigma}(\hat{q}) G_{\beta\nu}(\hat{k}-\hat{q}) \frac{dq}{(2\pi)^4} \]

\[ \Delta G_T^{-1}(\omega,k,k_3) = k_\alpha k_\beta \int U_{\alpha\beta}(\hat{q}) G_T(\hat{k}-\hat{q}) \frac{dq}{(2\pi)^4} \]

where \( U_{\mu\sigma}(\hat{q}) = 2Dq^{-3} G_{\alpha\mu}(\hat{q}) G^{*\beta\sigma}(\hat{q}) P_{\alpha\beta}(q) \)

The integrals are calculated using Distant Interaction Approximation \((k<<\Lambda)\) in which only the terms up to \( O((k/L)^2) \) are retained \( \rightarrow \) spectral gap

**Final result:** a coupled system of 4 nonlinear ordinary differential equations for all corrections. The system is solved analytically for weak and numerically for arbitrary stratification yielding expressions for scale-dependent horizontal and vertical eddy viscosities and eddy diffusivities
Theoretical results

Weak stable stratification

Expansion in powers of spectral Froude number,

\[ \mathcal{F} \approx 0.5(k/k_O)^{2/3} \]

\[ \nu_h/\nu_n = 1 + 0.095 \mathcal{F}^{-2} \]

\[ \nu_z/\nu_n = 1 - 0.31 \mathcal{F}^{-2} \]

\[ \kappa_h/\nu_n = \alpha + 0.054 \mathcal{F}^{-2} \]

\[ \kappa_z/\nu_n = \alpha - 0.4 \mathcal{F}^{-2} \]

\[ k_O = \sqrt{N^3/\varepsilon} \quad \text{– Ozmidov wavenumber} \]

\[ \alpha = \Pr_t^{-1} = \nu_n/\kappa_n = 0.72 \quad \text{– inverse turbulent Prandtl number} \]

\[ \nu_n, \kappa_n \quad \text{– eddy viscosity and diffusivity in neutral flows} \]

\[ \nu_n = 0.46 \varepsilon^{1/3} k^{-4/3} \]
Scale-dependent horizontal and vertical eddy viscosities and diffusivities

Figure 1: Normalized horizontal and vertical eddy viscosities and diffusivities as functions of $k/k_0$. Dashed vertical line indicates the maximum wave number threshold of internal wave generation in the presence of turbulence; $k_0 = (N^3/\varepsilon)^{1/2}$.
Some interpretations

For $k << k_O$ (strong stratification):

$$\nu_z(k) \propto \varepsilon^{1/3} k^{-4/3}$$
$$\kappa_z(k) \propto \varepsilon^{1/3} k^{-4/3}$$

$\implies$ a) Only scales up to $k_O^{-1}$ contribute to mixing of scalar, while the mixing of momentum continues on larger scales dominated by waves

b) $$\kappa_z \propto \left(\varepsilon/N^2\right)^{1/2}$$

c) The vertical turbulent Prandtl number is proportional to $Ri = N^2/S^2$

$$\frac{\nu_z}{\kappa_z} \propto Ri, \quad Ri > 1$$
**Turbulence spectra**

Due to anisotropy traditional 3D energy spectrum provides only limited information. Various 1D spectra are computed analytically for weak stratification:

\[
E_1(k_3) = \frac{8}{(2\pi)^4} \int U_{11}(\omega, k) d\omega dk_1 dk_2 = 0.626 \varepsilon^{2/3} k_3^{-5/3} + 0.214 N^2 k_3^{-3}
\]

Spectrum ~ \( k^{-3} \) is generated!

Transition from -5/3 to -3 spectrum at large scales; coefficients are in a good agreement with experimental data and LES (Carnevale et al, JFM, 2001).

The Gargett et al.(1981) -normalized spectrum of the vertical shear:

\[
\frac{E_S}{E_B} = \frac{2k_3^2 E_1(k_3) (\epsilon N)^{1/2}}{F(k_3/k_O) = 2 \times 0.626 (k_3/k_O)^{1/3} \left[1 + 0.34(k_3/k_O)^{-4/3}\right]}
\]

This scaling presents the normalized vertical shear spectrum as a universal function of \((k/k_O)\). **The QNSE theory provides rigorous theoretical basis for this universal scaling**
Temperature spectra

\[ E_T(k_1) = C_\theta\varepsilon_\theta \varepsilon^{-1/3} k_1^{-5/3} \left[ 1 + C_1 \left( \frac{k_1}{k_O} \right)^{-4/3} \right] + C_2 \left( \frac{d\Theta}{dz} \right)^2 k_1^{-3} \]

\[ E_T(k_3) = C_\theta\varepsilon_\theta \varepsilon^{-1/3} k_3^{-5/3} \left[ 1 + C_3 \left( \frac{k_3}{k_O} \right)^{-4/3} \right] + C_4 \left( \frac{d\Theta}{dz} \right)^2 k_3^{-3} \]

where \( C_\theta = 0.62 \) – Corrsin constant, \( C_1 = 0.068, \ C_2 = 0.27, \ C_3 = 0.41, \ C_4 = 0.13 \)

Assuming that for relatively strong stratification

\[ \varepsilon_\theta = 2 \Gamma \varepsilon (d\Theta/dz)^2 / N^2 \] where \( \Gamma \approx 0.3 \) is the mixing efficiency,

one gets,

\[ E_T(k_{1,3}) \approx C_\theta\varepsilon_\theta \varepsilon^{-1/3} k_{1,3}^{-5/3} + 0.3 \left( \frac{d\Theta}{dz} \right)^2 k_{1,3}^{-3} \]
The spectrum of the vertical shear of the horizontal velocity in the ocean; data from Gregg, Winkel, Sanford, JPO (1993). The theoretical prediction is shown by a gray line. This is the first time that these spectra are derived within an analytical theory.
Another example - Atmospheric spectra

Our theory predicts the vertical spectrum which is in a good agreement with the spectra observed in the stratosphere, troposphere, mesosphere, and thermosphere (our prediction is well approximated by the dashed line).
Other 1D spectra

\[ E_3(k_1) = \frac{8}{(2\pi)^4} \int U_{33}(\omega, k) d\omega dk_2 dk_3 = 0.626 \varepsilon^{2/3} k^{-5/3} - 0.704 N^2 k_3^{-3} \]

\[ E_3(k_3) = \frac{8}{(2\pi)^4} \int U_{33}(\omega, k) d\omega dk_1 dk_2 = 0.47 \varepsilon^{2/3} k^{-5/3} - 0.143 N^2 k_3^{-3} \]

Recall the vertical spectrum of horizontal velocity,

\[ E_1(k_3) = \frac{8}{(2\pi)^4} \int U_{11}(\omega, k) d\omega dk_1 dk_2 = 0.626 \varepsilon^{2/3} k_3^{-5/3} + 0.214 N^2 k_3^{-3} \]

The anisotropization manifests itself as energy increase in the horizontal velocity components at the expense of their vertical counterpart.
RANS modeling

Invoking energy balance equation, the eddy coefficients are recast in terms of Richardson number $Ri = N^2 / S^2$ or Froude number $Fr = \varepsilon / NK$

Figure 2: Normalized eddy viscosities and diffusivities as functions of $Ri$ and $Fr$.

- For $Ri > 0.1$, both vertical viscosity and diffusivity decrease, with the diffusivity decreasing faster than the viscosity (“residual” mixing due to effect of IGW?)
- Horizontal mixing increases with $Ri$. The model accounts for flow anisotropy.
- The crossover from neutral to stratified flow regime is replicated.
Comparison with data: $\text{Pr}_t$ as a function of $\text{Ri}$

Inverse Prandtl number, $\kappa_z/\nu_z$, as a function of $\text{Ri}$.

Dispersio relation for internal waves with turbulence

Complex poles yield the secular equation
\[ \det \left[ G^{-1}_{\alpha\beta}(\omega, k) \right] = 0 \]

Waves exist if the solution of the secular equation has a real part. Identifying this real part with the wave frequency \( \omega \) we obtain the dispersion relation

\[ \omega^2 = N^2 \sin^2 \theta \left\{ 1 - \left( \frac{k}{k_O} \right)^{4/3} \left[ \frac{\left( \frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n} \right) \cos^2 \theta + \left( \frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n} \right) \sin^2 \theta}{4 \sin \theta} \right]^2 \right\} \]

The limit of strong stratification \( \Rightarrow \) classical dispersion relation for linear waves, \( \omega = N \sin \theta \). Turbulence dominates at small scales. Criterion for wave generation is \( \omega^2 \geq 0 \) giving

\[ k_t = k_O \left| \frac{4 \sin \theta}{\left( \frac{\kappa_z}{\nu_n} - \frac{\nu_z}{\nu_n} \right) \cos^2 \theta + \left( \frac{\kappa_h}{\nu_n} - \frac{\nu_h}{\nu_n} \right) \sin^2 \theta} \right|^{3/2} \approx 32k_O | \sin \theta |^{3/2} \]
For implementation in PBL schemes, the theoretically derived stability functions, $\alpha_M = K_M/K_0$ and $\alpha_H = K_H/K_0$, were approximated by a fraction-polynomial fit ($K_0$ is the eddy viscosity at $Ri=0$):

$$\alpha_M = \frac{1 + 8 Ri^2}{1 + 2.3 Ri + 35 Ri^2}$$

$$\alpha_H = \frac{1.4 - 0.01 Ri + 1.29 Ri^2}{1 + 2.344 Ri + 19.8 Ri^2}$$

The fitting functions are valid for $Ri<1.5$. For larger values, flux Richardson number $R_f$ approaches limiting value $<0.5$.
**QNSE-based K-l model**

\[ \frac{\partial E}{\partial t} = K_M \left[ \left( \frac{\partial U}{\partial z} \right)^2 + \left( \frac{\partial V}{\partial z} \right)^2 \right] - \frac{g}{\Theta_0} K_H \frac{\partial \theta}{\partial z} - \varepsilon + \frac{\partial}{\partial z} \left( K_q \frac{\partial E}{\partial z} \right) \]

\[ \varepsilon = c_\varepsilon \frac{E^{3/2}}{l} ; \quad l_B = \frac{k_z}{k_z} \frac{K^{1/2}}{N} ; \quad l_N = c_N \frac{K^{1/2}}{N} ; \quad c_N = 0.75 \]

\[ \frac{1}{l} = \frac{1}{l_B} + \frac{1}{l_N} ; \quad c_\varepsilon = c_0^3, \quad c_0 = 0.55, \quad \lambda = B u_*/f \]

**---------------------------------------------**

\[ K_M = \alpha_M K_0, \quad K_H = \alpha_H K_0, \quad K_0 = c_0 l E^{1/2} \]
QNSE-based surface layer parameterization

Using theoretically derived stability functions $\alpha_M$, $\alpha_H$ and approximations of constant flux layer, we derived the drag coefficients for momentum and heat, $C_D$, $C_H$, that replace the Louis formulation. The corresponding expressions are:

\[
C_D = \frac{\kappa^2}{\left(\ln \frac{z}{z_0} + \psi_M(\zeta) - \psi_M(\zeta_0)\right)^2}, \quad \zeta = z/L, \quad \zeta_0 = z_0/L
\]

\[
C_H = \frac{\kappa^2}{\left(\ln \frac{z}{z_0} + \psi_M(\zeta) - \psi_M(\zeta_{0T})\right)} \left(\Pr_0 \ln \frac{z}{z_{0T}} + \psi_H(\zeta) - \psi_H(\zeta_{0T})\right)
\]

\[
\psi_M(\zeta) = 2.25\zeta - 0.2\zeta^2
\]

\[
\psi_H(\zeta) = 2\Pr_0 \zeta + 0.1((\zeta - 0.5)^5 - 0.5^5),
\]

$\Pr_0 = 0.71$ – turbulent Prandtl number for neutral flow
Testing of the model - neutral ABL

Comparison with Leipzig wind profile
Testing in WRF – BASE temperature profiles

Comparison with LES results (Stroll and Porte-Agel, BLM, 2008)
Testing in WRF – BASE velocity profiles

Comparison with LES results (Stroll and Porte-Agel, BLM, 2008)
Effect of surface layer parameterization

QNSE surface scheme eliminates warm temperature bias. With other schemes, increase in resolution may decrease the bias but does not eliminate it.
Unstable stratification (Convection)
Conclusions

- Derivation of the QNSE model of turbulence is maximally proximate to first principles
- Theory explicitly resolves horizontal-vertical anisotropy
- Accounts for the combined effect of turbulence and waves
- Predicts correct behavior of $Pr_t$ as a function of $Ri$
- Anticipates the absence of the critical $Ri$
- Yields modification of the classical dispersion relation for internal waves that accounts for turbulence
- Yields analytical expressions for various 1D and 3D spectra; captures transition from the $-5/3$ to the $N^2k_z^{-3}$ vertical spectrum of the horizontal velocity and recovers Gargett et al. scaling
- Provides subgridscale closures for both LES and RANS
- The QNSE theory has been implemented in K-$\varepsilon$ and K-$\ell$ models of stratified ABL
- Good agreement with CASES-99 and other data sets has been found for cases selected for the GABLS 2 experiment
- The new stability functions improve predictive skills of HIRLAM in +24h and +48h weather forecasts
- Is being incorporated in WRF (Weather Research and Forecasting) – a new model developed at NCAR. Will become a standard module of WRF
Acknowledgments

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References


Validation of the QNSE theory in modeling of atmospheric boundary layers and numerical weather prediction

- Validation was conducted for models in both K-\(\varepsilon\) and K-\(\ell\) format
- Data from numerous observation campaigns was employed

**CASES99**

Data-asterisks, model - lines

![Graphs showing Potential temperature and Wind speed](image)